MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN2B. Gaussian Elimination and Identification of Solutions.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights parts of the text that are used in the lecture sessions.

Theorem 2B.1. Gaussian Elimination

Let $\mathbf{Ax} = \mathbf{b}$ be a matrix equation and let $\mathbf{M} \in \mathbb{R}^{m \times n}$ be its corresponding augmented matrix. We describe the **Gaussian Elimination** algorithm below. We iterate over the rows $\mathbf{R_1}, ..., \mathbf{R_m}$ of $\mathbf{M}$ with index $\mathbf{i}$.

(a) If $\mathsf{Pivot}\,\mathbf{R_i}$ is defined (i.e. $\mathbf{R_i}$ is not a zero row):

- **(i)** For convenience, we may perform a Type (I) operation (swapping rows) to replace $\mathsf{Pivot}\,\mathbf{R_i}$ with one of the entries below $\mathsf{Pivot}\,\mathbf{R_i}$. We sometimes do this to choose the smallest to simplify calculations. (In algorithms, the largest number is usually chosen)
- **(ii)** Use row operations of Type (II) and/or Type (III) to eliminate the entries below $\mathsf{Pivot}\,\mathbf{R_i}$ (i.e. make the entries equal zero).

(b) If $\mathsf{Pivot}\,\mathbf{R_i}$ is not defined (i.e. the row $\mathbf{R_i}$ is a zero row), do row operations of Type (I) such that $\mathbf{R_i}$ such that the all zero rows are the bottom rows of $\mathbf{M}$ and $\mathsf{Pivot}\,\mathbf{R_i}$ are in strictly increasing order.

This results in $\mathbf{M}$ in row echelon form. To achieve reduced row echelon form:

(c) Perform a Type (I) row operation by multiplying the all rows with pivots by the reciprocal of the pivot. Then, for each pivot, from bottom to top, we use Type (III) row operations to eliminate the values above each pivot.

It is not strictly necessary to do the algorithm stated above as long as we achieve row echelon form or reduced echelon form. I've added the algorithm as a starting point if needed.

We will present ways to interpret and use the row reduced augmented matrix. However, we introduce some notation and terminology to help with the definitions and explanations.

Definition 2B.2. Consistent and Inconsistent Systems

Let $\mathbf{Ax} = \mathbf{b}$ be a matrix equation for a linear system of equations. Then,

(a) We say that the system is **consistent** if there exists at least one solution to the system.

(b) We say that the system is **inconsistent** if the system has no solutions. Equivalently, the system is **inconsistent** if and only if it is not consistent.

The definitions above are sometimes used in other resources so we introduce them. Then, we proceed to more definitions.

Definition 2B.3. Row and Column Notation

Let $\mathbf{M} \in \mathbb{R}^{m \times n}$ be some matrix. Then,

(a) Denote the i^{th} row of $\mathbf{M}$ as $\mathsf{row}_i\,\mathbf{M}$. Observe that $\mathsf{row}_i\,\mathbf{M}$ is a row vector with n entries.

(b) Denote the j^{th} column of $\mathbf{M}$ as $\mathsf{col}_j\,\mathbf{M}$. Observe that $\mathsf{col}_j\,\mathbf{M}$ is a column vector with m entries.

Definition 2B.4. Basic and Free Variables

Let $\mathbf{M} \in \mathbb{R}^{m\times(n+1)}$ be the **row reduced** augmented matrix for a system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Recall that by matrix multiplication, each unknown x_i in $\mathbf{x}$ corresponds to a column in the matrix $\mathbf{A}$ and therefore, corresponds to a column in the row reduced augmented matrix $\mathbf{M}$. Then,

(a) The unknown x_i is a **basic variable** if and only if the column of $\mathsf{col}_i\,\mathbf{M}$ contains a pivot.

(b) The unknown x_i is a **free variable** if and only if the column of $\mathsf{col}_i\,\mathbf{M}$ does **not** contain a pivot.

Now, we proceed for the interpretation of the row reduced augmented matrix, i.e. identification of the solution set using the row reduced augmented matrix.

Theorem 2B.5. Cardinality of Solutions from the Row Reduced Augmented Matrix

Let $\mathbf{M} \in \mathbb{R}^{m\times(n+1)}$ be a row reduced augmented matrix for some system $\mathbf{Ax} = \mathbf{b}$ with m equations and n unknowns. Then,

(a) The system is **inconsistent** (i.e. has no solutions) if and only if the last column of $\mathbf{M}$ contains a pivot. This corresponds to having a row with zeros in the first n entries and some nonzero constant for the m^{th} entry.

(b) The system has **exactly one solution** if and only if there are no free variables. The solution set, as it sits in $\mathbb{R}^n$, is a point.

(c) The system has **infinitely many solutions** if and only if the system has free variables.

The number of free variables determine the shape of the solution set as it sits in $\mathbb{R}^n$. e.g. One free variable corresponds to the solution set being a line; Two free variables correspond to the solution set being a plane.

If the system is consistent, we can identify the solution set the solution set can be expressed in terms of the free variables of the system.

Definition 2B.6. Back-Substitution

Let $\mathbf{M} \in \mathbb{R}^{m\times(n+1)}$ be the **row reduced** augmented matrix for a consistent system $\mathbf{Ax} = \mathbf{b}$ with m equations and n unknowns. The method of **back-substitution** involves the following:

(a) Transform the row reduced augmented matrix into a linear system of equations.

(b1) If there are no free variables, then the $\mathbf{x}$ is equal to some constant vector. This can be identified either by proceeding to the reduced row echelon form or by substitution/elimination on the linear system of equations.

Note: Both options correspond to equivalent operations

(b2) If there are free variables, express all basic variables in terms of the free variables. i.e. express all the unknowns/variables as functions of the free variables. Note that this step is much easier if the augmented matrix is in reduced row echelon form.

The unknown $\mathbf{x}$ as a function of the free variables determines the solution set of the system $\mathbf{Ax} = \mathbf{b}$ as a linear combinations of some set of vectors, one for each free variable, added to some vector for translation.

We then proceed to one application.

Definition 2B.7. Notation for Standard Basis Vectors

We define the **standard basis vectors** of $\mathbb{R}^n$ as the set of n vectors denoted as $\mathbf{e_1}, ..., \mathbf{e_n}$ such that for each i, the vector $\mathbf{e_i}$ has 1 for the i^{th} entry and zeros everywhere else.

Equivalently, the standard basis vectors of $\mathbb{R}^n$ correspond the columns of the identity matrix $\mathbf{I_n}$ with $\mathbf{e_i}$ referring to the column $\mathsf{col}_i\, \mathbf{I_n}$.

Theorem 2B.8. Finding Inverses

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix. Then, the problem of determining the invertibility of $\mathbf{A}$ is equivalent to finding solutions $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}$ for the systems $\mathbf{Ax_1} = \mathbf{e_1}$, $\mathbf{Ax_2} = \mathbf{e_2}, ..., \mathbf{Ax_2} = \mathbf{e_2}$. We can simultaneously solve all equations using the matrix equation:

$$\mathbf{A}\begin{pmatrix} | & | & & | \\ \mathbf{x_1} & \mathbf{x_2} & \cdots & \mathbf{x_n} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \mathbf{e_1} & \mathbf{e_2} & \cdots & \mathbf{e_n} \\ | & | & & | \end{pmatrix} = \mathbf{I_n} \quad \overset{\text{with}}{\underset{\text{matrix}}{\text{augmented}}} \quad \mathbf{M} = \begin{pmatrix} | & | & | & & | \\ \mathbf{A} & \mathbf{e_1} & \mathbf{e_2} & \cdots & \mathbf{e_n} \\ | & | & | & & | \end{pmatrix}$$

Then, $\mathbf{A}$ is invertible if and only if there exists exactly one solution to the matrix equation above (or equivalently, there are no free variables in the row reduced augmented matrix) and the inverse $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \begin{pmatrix} | & | & & | \\ \mathbf{x_1} & \mathbf{x_2} & \cdots & \mathbf{x_n} \\ | & | & & | \end{pmatrix}$$